LESSON 8-2 EXPONENTIAL GROWTH AND DECAY

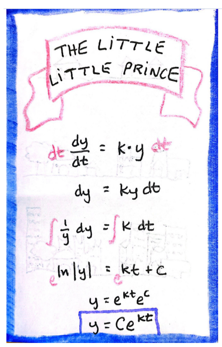
Mathematical models in which the rate of change of a variable is proportional to the variable itself are common in both the business and scientific worlds.

Suppose that the rate of change of y (with respect to time) is proportional to y itself.

$$\frac{dy}{dt} \quad = \quad k \quad \bullet \quad y$$

Rate of change of y with respect to time	=	constant of proportionality	$\bullet$	amount of substance y present at time t (y is a function of t)

Example 1: Separate variables and solve the differential equation above.



ending amount
starting amount
rate
time

The equation from Example 1 is called the **Basic Law of Exponential Growth or Decay**:

$$y = Ce^{kt}$$

Constants:

C is the initial value (the amount of substance present at time $t = 0$)

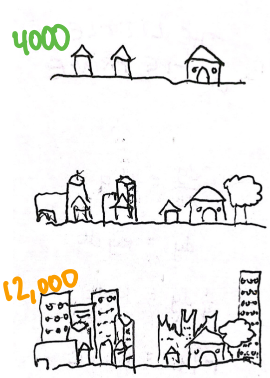
k is the constant of proportionality ($k > 0$ for growth and $k < 0$ for decay)

Variables:

t is the variable for time

y is the amount of substance present at time t. (y is a function of t.)

Example 2: What is the rate of growth of the population in a city whose population triples every 100 years? Assume that the population growth can be modeled by the Basic Law of Exponential Growth, and express your answer as a percent (rounded to the nearest hundredth of a percent).



$y = Ce^{kt}$

12,000

4000

how much it grows in one year

100

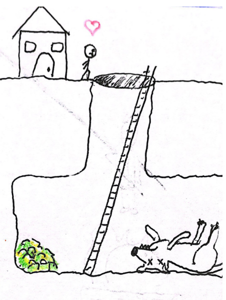
$$12{,}000 = 4000e^{k(100)}$$

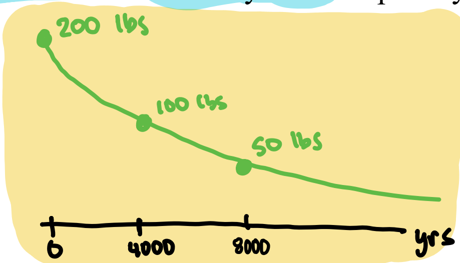
$$3 = e^{k(100)}$$

$$\ln 3 = k(100)$$

$$k = \frac{\ln 3}{100} = 0.010986 \approx 1.10\%$$

Example 3: Let y represent the mass, in pounds, of a radioactive element whose half-life is 4000 years. If there are 200 pounds of the element in an inactive mine, how much will still remain in 1000 years? Express your answer to 3 or more decimal place accuracy.





$100 = 200e^{k(4000)}$

$0.5 = e^{k(4000)}$

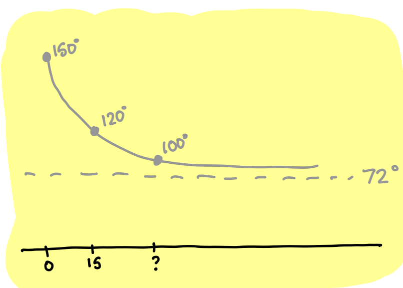
$\ln 0.5 = k(4000)$

$k = -0.000173286795$

$y = 200e^{-0.000173286795(1000)}$

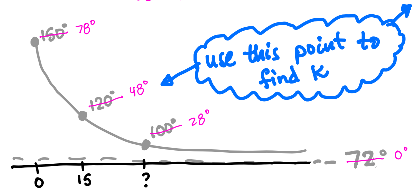
$\boxed{\approx 168.179 \text{ years}}$

Example 4: Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference between the temperature of the object and the temperature of its surroundings. Suppose a metal figurine, heated to $150°$ F, is brought into a room having a constant temperature of $72°$ F. If the figurine cools from $150°$ to $120°$ in 15 minutes, how long will it take for the figurine to reach a temperature of $100°$ F? Let $y =$ temperature, and express your answers to the nearest minute.



$y = Ce^{kt}$ only works for asymptotes at $y = 0$

shift everything down 72°



use this point to find k

$48 = 78e^{k(15)}$

$k = -0.0323671877$

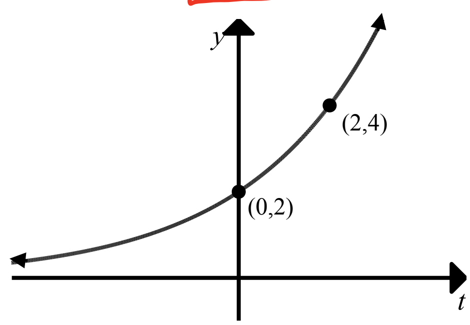
use this point to find t

$28 = 78e^{-0.0323671877t}$

$\boxed{t = 31.653 \text{ minutes}}$

ASSIGNMENT 8-2

1. Find the particular equation of the form $y = Ce^{kt}$ which represents the exponential growth graph shown at right. You must first solve for C. Then, you must solve for k. Finally, you can write the equation in the form $y = Ce^{kt}$ using your values for C and k.



2. \$1000 is placed into a certificate of deposit (CD) in which interest is compounded continuously at a rate of $5\frac{1}{2}\%$ per year (actual rate of return will be higher due to compounding of interest). Use your calculator and the formula $A = Pe^{rt}$ to find:

   a. the amount that the CD would be worth in 1 year. 5 years. 10 years.
   b. the time it would take the CD to be worth \$1,200.
   c. the time it would take the CD to double in value.