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BC Topic 8 — Taylor Series
due Tuesday, November 28

Taylor Series

In this section you will be finding polynomial functions that can be used to approximate transcendental functions. If $P(x)$ is a polynomial function used to approximate some other function $f(x)$, they must contain the same point with some x-value c. That means $P(c)=f(c)$. To be a better approximation they should have the same slope at that point. This means $P'(c)=f'(c)$. For even greater accuracy, $P''(c)=f''(c)$ and so on. Putting this together gives us the **Taylor Polynomial Expansion:** If $f(x)$ has derivatives of all orders it can be approximated by the polynomial function shown.

$$\boxed{P_n(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n}$$

This is called an n^{th} degree or n^{th} order Taylor Polynomial centered at c or expanded about c.

When the center is at $c=0$ the Taylor polynomial is called a **Maclaurin Polynomial** which can be written as :

$$\boxed{P_n(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n}$$

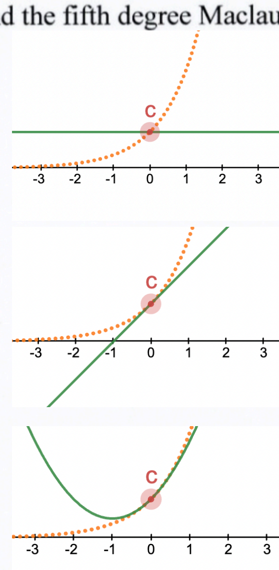
Example 1. Use the definition of a Maclaurin Polynomial to find the fifth degree Maclaurin Polynomial for $f(x)=e^x$.

$f(x)=e^x$

Type these into Desmos to crank out these numbers.

The TI calculators are tedious with the higher derivatives.

$f(0)=1$	starting height	1
$f'(0)=1$	starting slope	$1+1x$
$f''(0)=1$	starting concavity	$1+1x+\frac{1}{2!}x^2$
$f'''(0)=1$	starting jerk	
$f^{(4)}(0)=1$	starting snap	
$f^{(5)}(0)=1$	starting crackle	



$$P_5(x)=1+1x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5$$
$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}$$

This polynomial is a good approximation for $f(x)=e^x$. By extending the pattern into an infinite series it becomes exactly correct instead of an approximation.

$$f(x)=e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$$

The general form of Taylor and Maclaurin Polynomials can be extended to Taylor and Maclaurin Series.

Taylor Series (provided $f(x)$ has derivatives of all orders)

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots$$

Maclaurin Series

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

These formulas allow us to form a power series for functions that cannot be written as geometric power series.

Example 2. Use the definition of a Taylor Polynomial to find the fourth order Taylor Polynomial and the Taylor Series for $f(x)=\cos x$ centered at $c=\pi$.

$f(x)=\cos x$

$f(\pi)=-1$ — starting height

$f'(\pi)=0$ — starting slope

$f''(\pi)=1$ — starting concavity

$f'''(\pi)=0$ — starting jerk

$f^{(4)}(\pi)=-1$ — starting snap

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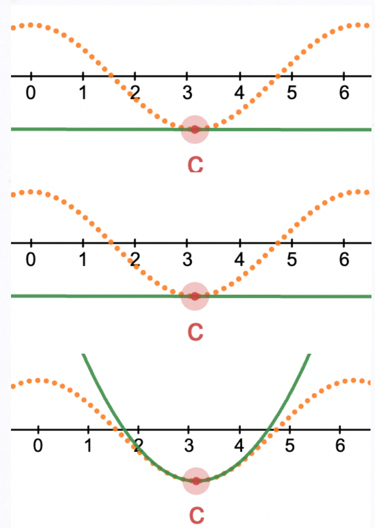

$$P_4(x)=-1+0(x-\pi)+\frac{1}{2!}(x-\pi)^2+\frac{0}{3!}(x-\pi)^3+\frac{-1}{4!}(x-\pi)^4$$

$$f(x)=-1+\frac{(x-\pi)^2}{2!}-\frac{(x-\pi)^4}{4!}+\frac{(x-\pi)^6}{6!}+\cdots$$

Example 3. Use your Taylor Polynomial from example 2 to approximate $\cos 3$.

$\cos 3\approx -1+\frac{(3-\pi)^2}{2!}-\frac{(3-\pi)^4}{4!}=-.989857$ Note: $\cos 3=-.98999$

Example 4. Use the definition of a Maclaurin Series to find the Maclaurin series for $f(x)=\sin x$.

$f(x)=\sin x$

$f(0)=0$

$f'(0)=1$

$f''(0)=0$

$f'''(0)=-1$

$f^{(4)}(0)=0$

$f^{(5)}(0)=1$

Type these into Desmos to crank out these numbers.

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$$f(x)=\sin x = 0 + 1x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} + \cdots$$

Example 6. Find a power series for $f(x)$ centered at $c=1$ if $f(1)=2$ and $f^{(n)}(1)=n!$.

$f'(1)=1!=1$

$f''(1)=2!$

$f'''(1)=3!$

etc

$$f(x)=2+1(x-1)+\frac{2!}{2!}(x-1)^2+\frac{3!}{3!}(x-1)^3+\cdots$$

$$=2+\sum_{n=1}^{\infty}(x-1)^n$$

Assignment

1. Use the definition to find a fourth degree Maclaurin Polynomial for $f(x)=\frac{1}{e^x}$.
2. Use the polynomial from Problem 1 to approximate $\frac{1}{\sqrt{e}}$.
3. Use the definition to find a fourth degree Maclaurin Polynomial for $f(x)=e^{3x}$.
4. Use the definition to find a fifth degree Maclaurin Polynomial for $g(x)=xe^x$.
5. Use the definition to find a third degree Taylor Polynomial centered at $c=1$ for $f(x)=\sqrt[3]{x}$.
6. Use the definition to find a fourth degree Taylor Polynomial centered at $c=1$ for $h(x)=\ln x$.
7. Use the polynomial from Problem 6 to approximate $\ln 1.3$.
8. Use the definition to find a Taylor Series centered at $c=\frac{\pi}{4}$ for $f(x)=\sin x$.
9. Use the definition to find a Taylor Series centered at $c=0$ for $f(x)=\cos(2x)$. Show four terms (Zero terms don't count.) and a general term.
10. Use the definition to write a Maclaurin Series for $f(x)=\frac{1}{1+x}$. Show four terms and the general term.
11. Write a geometric series expansion for $f(x)=\frac{1}{1+x}$. Also give the interval of convergence.
12. Write four terms and the general term of the Taylor series expansion of $f(x)=\frac{1}{x-1}$ about $x=2$.

14. The Taylor Series of a function about $x=3$ is given by
$f(x)=1+\frac{3(x-3)}{1!}+\frac{5(x-3)^2}{2!}+\frac{7(x-3)^3}{3!}+\cdots+\frac{(2n+1)(x-3)^n}{n!}+\cdots$ What is the value of $f'''(3)$?

15. Let $f(x)$ be a function such that $f(0)=2$, $f'(x)=3f(x)$, and the n^{th} derivative of f is given by $f^{(n)}(x)=3f^{(n-1)}(x)$.

(a) Give the first four terms and the general term of the Taylor Series for f centered at $x=0$.

18. The Maclaurin series for $f(x)$ is given by $1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots+\frac{x^n}{(n+1)!}+\cdots$

(a) Find $f'(0)$. (b) Find $f^{(15)}(0)$.

19. Find a geometric power series for $f(x)=\frac{x}{1+4x^2}$. Show four terms and the general term. Also give the series using sigma notation and give the interval of convergence.

20. Find a function for the geometric power series $f(x)=\sum_{n=0}^{\infty}4(3x)^n$. Also give the interval of convergence.

Determine whether the following series converge or diverge. Find the sum when possible.

21. $\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$ 22. $\sum_{n=0}^{\infty}(1.3)^n$ 23. $\sum_{n=0}^{\infty}\frac{n^2-1}{n+1}$

Selected Answers:

1. $f(x)\approx 1-x+\frac{1}{2}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4$ 2. .60677 3. $f(x)\approx 1+3x+\frac{9}{2}x^2+\frac{27}{3!}x^3+\frac{81}{4!}x^4$

4. $g(x)\approx x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4+\frac{1}{24}x^5$ 5. $f(x)\approx 1+\frac{1}{3}(x-1)-\frac{1}{9}(x-1)^2+\frac{10}{27\bullet 3!}(x-1)^3$

7. .261975 8. $f(x)=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right)-\frac{1}{\sqrt{2}}\bullet\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2-\frac{1}{\sqrt{2}}\bullet\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3+\frac{1}{\sqrt{2}}\bullet\frac{1}{4!}\left(x-\frac{\pi}{4}\right)^4+\cdots$

10. $f(x)=1-x+x^2-x^3+\cdots+(-1)^n x^n+\cdots$

12. $f(x)=1-(x-2)+(x-2)^2-(x-2)^3+\cdots+(-1)^n(x-2)^n+\cdots$

14. $f'''(3)=7$ 15a. $f(x)=2+6x+9x^2+9x^3+\cdots+\frac{2(3^n)}{n!}x^n+\cdots$

18a. $f'(0)=\frac{1}{2}$ b. $f^{(15)}(0)=\frac{1}{16}$

19. $f(x)=x-4x^3+16x^5-64x^7+\cdots+(-1)^n4^nx^{2n+1}+\cdots=\sum_{n=0}^{\infty}(-1)^n4^nx^{2n+1}$, $-\frac{1}{2}<x<\frac{1}{2}$

21. converges to 3 23. diverges