name:

BC Topic 7 — Power Series
due Monday, November 13

Series:

$$1 + 0.1 + 0.01 + 0.001 + 0.0001 + \dots$$

$$5 + 7 + 9 + 11 + 13 + 15 + 17 + \dots$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

$$3 - \frac{3}{2} + \frac{3}{3} - \frac{3}{4} + \frac{3}{5} - \frac{3}{6} + \frac{3}{7} - \dots$$

Power Series:

$$1 + 0.1x + 0.01x^2 + 0.001x^3 + 0.0001x^4 + \dots$$

$$5 + 7x^2 + 9x^4 + 11x^6 + 13x^8 + 15x^{10} + \dots$$

$$1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \frac{1}{32}x^5 + \dots$$

$$3x - \frac{3}{2}x^3 + \frac{3}{3}x^5 - \frac{3}{4}x^7 + \frac{3}{5}x^9 - \frac{3}{6}x^{11} + \frac{3}{7}x^{13} - \dots$$

You can think of a power series as an infinite polynomial.

THE COOL PART

You can mimic any curve with an infinite polynomial (a power series).

EXAMPLE

Graph this rational function on Desmos. $f(x) = \dfrac{1}{1+x}$

Next, graph this 9th-degree polynomial. $f(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - x^9$

Finally, graph this 100th-degree polynomial. $f(x) = \sum_{n=0}^{100} 1(-x)^n$

If you were to graph the entire power series (to infinity), you would see it perfectly line up with the original rational function for the interval $-1 < x < 1$. This is called the *interval of convergence*.

Power Series, Geometric Power Series

A series with variable terms like $1+x+x^2+x^3+\cdots+x^n+\cdots$ is called a **power series.** Note that this series is a **geometric power series.** If it converges, what must be true about the variable x?

For these x-values $1+x+x^2+x^3+\cdots+x^n+\cdots=$ $\frac{a}{1-r} = \sum_{n=0}^{\infty} a(r)^n$

This means for these x-values, the function $f(x)=\dfrac{1}{1-x}$ can be written as $f(x)=\sum_{n=0}^{\infty} x^n$.

Examples: Find a power series for each of the following functions. Show four terms and the general term. Also give the series using sigma notation and give the interval of convergence.

1. $f(x)=\dfrac{1}{1+x} = \dfrac{1}{1-(-x)}$ $a=1$, $r=-x$

$f(x) = 1 - x + x^2 - x^3 + \cdots + 1(-x)^n$

$f(x) = \sum_{n=0}^{\infty} a(r)^n = \sum_{n=0}^{\infty} 1(-x)^n$ converges when $-1 < -x < 1$; $1 > x > -1$; IOC: $-1 < x < 1$

2. $g(x)=\dfrac{3}{1-2x}$ $a=3$, $r=2x$

$g(x) = 3 + 6x + 12x^2 + 24x^3 + \cdots + 3(2x)^n + \cdots$

$g(x) = \sum_{n=0}^{\infty} 3(2x)^n$ converges when $-1 < 2x < 1$; IOC: $-\frac{1}{2} < x < \frac{1}{2}$

3. $h(x)=\dfrac{1}{3x} = \dfrac{1}{1-(-3x+1)}$ $a=1$, $r=-3x+1$

$h(x) = 1 + (-3x+1) + (-3x+1)^2 + \cdots + (-3x+1)^n + \cdots$

$h(x) = \sum_{n=0}^{\infty} 1(-3x+1)^n$ converges when $|r| < 1$: $-1 < -3x+1 < 1$; $-2 < -3x < 0$; IOC: $\frac{2}{3} > x > 0$

Power Series by Substitution:

Examples:

4. Using the power series from Example 1, make a new power series for $f(x^2)$.

$f(x) = 1 - x + x^2 - x^3 + \cdots + 1(-x)^n$

$f(x^2) = 1 - (x^2) + (x^2)^2 - (x^2)^3 + \cdots$

$f(x^2) = 1 - x^2 + x^4 - x^6 + \cdots$

5. Using the power series from Example 2, make a new power series for $g(\sqrt{x})$.

$g(x) = 3 + 6x + 12x^2 + 24x^3 + \cdots + 3(2x)^n + \cdots$

$g(\sqrt{x}) = 3 + 6(\sqrt{x}) + 12(\sqrt{x})^2 + 24(\sqrt{x})^3 + \cdots$

$g(\sqrt{x}) = 3 + 6x^{1/2} + 12x + 24x^{3/2} + \cdots$

Find a geometric power series for each of the following functions. Show four terms and the general term. Also give the series using sigma notation and give the interval of convergence.

1. $\frac{1}{1-3x}$

2. $\frac{2}{1-x^3}$

3. $\frac{x}{1+x}$

4. $\frac{1}{1+(-x-3)}$

5. $\frac{3}{4x}$

6. $\frac{1}{2-2x}$

Find a function for each of the following geometric power series. Also give the interval of convergence.

7. $\sum_{n=0}^{\infty}(2x)^n$

8. $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n x^n$

9. $\sum_{n=0}^{\infty}4(x-1)^n$

10. $\sum_{n=1}^{\infty}\left(x^2\right)^n$

11. $\sum_{n=0}^{\infty}(\sin x)^n$

14. Find a geometric power series for $g(x)=\frac{1}{1+x}$. Show four terms and the general term.

15. Use the answer to Problem 14 to find a power series for $\frac{1}{1+x^2}$.

Use the function $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$ to find the following. Answer using sigma notation.

17. $f(-x)$

18. $f'(x)$

Use the function $g(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$ to find the following. Show four terms and the general term.

20. $g\left(\sqrt{x}\right)$

21. $g'(x)$

1. $1+3x+9x^2+27x^3+\cdots+(3x)^n+\cdots=\sum_{n=0}^{\infty}(3x)^n$; $\left(-\frac{1}{3},\frac{1}{3}\right)$

2. $2+2x^3+2x^6+2x^9+\cdots+2\left(x^3\right)^n+\cdots=\sum_{n=0}^{\infty}2\left(x^3\right)^n$; $(-1,1)$

3. $x-x^2+x^3-x^4+\cdots+(-1)^n x^{n+1}+\cdots=\sum_{n=0}^{\infty}(-1)^n x^{n+1}=\sum_{n=1}^{\infty}(-1)^{n+1}x^n$; $(-1,1)$

5. $3+3(1-4x)+3(1-4x)^2+3(1-4x)^3+\cdots+3(1-4x)^n+\cdots=\sum_{n=0}^{\infty}3(1-4x)^n$; $\left(0,\frac{1}{2}\right)$

6. $\frac{1}{2}+\frac{1}{2}x+\frac{1}{2}x^2+\frac{1}{2}x^3+\cdots+\frac{1}{2}x^n+\cdots=\sum_{n=0}^{\infty}\frac{1}{2}x^n$; $(-1,1)$

7. $\frac{1}{1-2x}$; $\left(-\frac{1}{2},\frac{1}{2}\right)$

8. $\frac{1}{1+\frac{1}{2}x}=\frac{2}{2+x}$; $(-2,2)$

9. $\frac{4}{1-(x-1)}=\frac{4}{2-x}$; $(0,2)$

10. $\frac{x^2}{1-x^2}$; $(-1,1)$

14. $1-x+x^2-x^3+\cdots+(-1)^n x^n+\cdots$

15. $1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}+\cdots$

17. $f(-x)=\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}$

18. $f'(x)=\sum_{n=1}^{\infty}x^{n-1}$

20. $g\left(\sqrt{x}\right)=1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+\cdots+\frac{(-1)^n x^n}{(2n)!}+\cdots$

21. $g'(x)=-x+\frac{x^3}{3!}-\frac{x^5}{5!}+\cdots+\frac{(-1)^n x^{2n-1}}{(2n-1)!}+\cdots$