name:

BC Topic 6 — Ratio Test
due Wednesday, November 1

Ratio Test: (useful for series involving <u>factorials</u> or <u>exponentials</u>)

1. $\sum a_n$ <u>converges</u> if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|<1$.
2. $\sum a_n$ <u>diverges</u> if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|>1$ or $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\infty$.
3. The Ratio Test is inconclusive if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=1$

Examples: Determine the convergence or divergence.

1. $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{3^n}$

$$\lim_{n\to\infty}\left|\frac{(n+1)!/3^{n+1}}{n!/3^n}\right| =$$

$$\lim_{n\to\infty}\left(\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!}\right)=$$

$$\lim_{n\to\infty}\frac{n+1}{3}=\infty$$

The series diverges by the RT.
(nTT would also work)

2. $\displaystyle\sum_{n=1}^{\infty}\frac{3^{n+1}}{4^n n^2}$

$$\lim_{n\to\infty}\left(\frac{3^{n+2}}{4^{n+1}(n+1)^2}\cdot\frac{4^n n^2}{3^{n+1}}\right)=$$

$$\lim_{n\to\infty}\frac{3n^2}{4(n+1)^2}=$$

$$\frac{3}{4}<1$$

The series conv. by RT

3. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n\sqrt{n}}{n-1}$

$$\cancel{\lim_{n\to\infty}\frac{\sqrt{n+1}}{n}\cdot\frac{n-1}{\sqrt{n}}=1}$$

~~RT is inconclusive~~

$$\lim_{n\to\infty}\frac{\sqrt{n}}{n-1}=0$$

$\frac{\sqrt{n}}{n-1}$ is decr.
conv. by AST

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(2n+1)!!}{3^n(2n-1)n!}=\sum_{n=1}^{\infty}\frac{(2n+1)(2n-1)(2n-3)\cdots5\cdot3\cdot1}{3^n(2n-1)n!}$

$$\lim_{n\to\infty}\left(\frac{(2n+3)!!}{3^{n+1}(2n+1)(n+1)!}\cdot\frac{3^n(2n-1)n!}{(2n+1)!!}\right)$$

$$\lim_{n\to\infty}\frac{(2n+3)\ (2n-1)}{3\ (n+1)(2n+1)}$$

$$\lim_{n\to\infty}\frac{4n^2}{6n^2}$$

$$=\frac{2}{3}<1$$

conv. by R.T.

Convergence/Divergence Tests

n^{th} term test	div. if $\lim\limits_{n\to\infty} a_n \neq 0$ (cannot be used to show convergence)
Geom. series test	$\sum\limits_{n=0}^{\infty} ar^n$ $\quad \lvert r\rvert<1 \to$ conv. , $\lvert r\rvert \geq 1 \to$ div. , $\mathrm{S}=\dfrac{\mathrm{a}}{1-r}$
p-series	$\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ $\quad p>1\to$ conv. , $p\leq 1 \to$ div.
Alternating series	decr. terms and $\lim\limits_{n\to\infty} a_n = 0 \to$ conv.
Ratio test	$\lim\limits_{n\to\infty}\left\lvert\dfrac{a_{n+1}}{a_n}\right\rvert<1\to$ conv. , $\lim\limits_{n\to\infty}\left\lvert\dfrac{a_{n+1}}{a_n}\right\rvert>1\to$ div. , (inconclusive if $\lim\limits_{n\to\infty}\left\lvert\dfrac{a_{n+1}}{a_n}\right\rvert=1$) (works well for factorials and exponentials)
Direct Comparison	a series with terms **smaller** than a known convergent series also converges a series with terms **larger** than a known divergent series also diverges

Mixed Examples: Determine the convergence or divergence.

5. $\sum\limits_{n=1}^{\infty} \dfrac{n-1}{2n+1}$

$\lim\limits_{n\to\infty} \dfrac{n-1}{2n+1} = \dfrac{1}{2} \neq 0$

div. by nTT

7. $\sum\limits_{n=1}^{\infty} \left(\dfrac{e}{3}\right)^n$

$r = \dfrac{e}{3} < 1$

The series conv. by GST

9. $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{3}{n^2}$

$\lim\limits_{n\to\infty} \dfrac{3}{n^2} = 0$

The terms dec. in abs. val.

The series conv. by AST

10. $\dfrac{1}{10}+\dfrac{1\cdot 2}{10^2}+\dfrac{1\cdot 2\cdot 3}{10^3}+\dfrac{1\cdot 2\cdot 3\cdot 4}{10^4}+\dfrac{1\cdot 2\cdot 3\cdot 4\cdot 5}{10^5}+\cdots$

$\sum\limits_{n=1}^{\infty} \dfrac{n!}{10^n}$

$\lim\limits_{n\to\infty} \left(\dfrac{(n+1)!}{10^{n+1}} \cdot \dfrac{10^n}{n!}\right)$

$\lim\limits_{n\to\infty} \dfrac{n+1}{10} = \infty$ Div. by RT

Use the ratio test to determine convergence or divergence if possible.

1. $\sum_{n=1}^{\infty} \frac{n!}{n^3}$ 2. $\sum_{n=0}^{\infty} n\left(\frac{2}{3}\right)^n$ 3. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ 4. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

5. $\sum_{n=1}^{\infty} \frac{(2n)!}{n3^n}$ 6. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ 7. $\sum_{n=0}^{\infty} \frac{(2n+1)!!}{n!} = \sum_{n=1}^{\infty} \frac{(2n+1)(2n-1)(2n-3)\cdots 3\cdot 1}{n(n-1)(n-2)\cdots 2\cdot 1}$

Determine convergence or divergence using any test.

8. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{n}$ 9. $\sum_{n=1}^{\infty} \frac{3}{n}$ 10. $\sum_{n=0}^{\infty} \frac{1}{3^n}$ 11. $\sum_{n=1}^{\infty} \frac{4}{n\sqrt{n}}$

12. $\sum_{n=1}^{\infty} (-1)^n \frac{3n}{n+1}$ 14. $\sum_{n=2}^{\infty} \frac{2^n}{\ln n}$ 15. $\sum_{n=1}^{\infty} \frac{|\cos n|}{4^n}$

16. $\sum_{n=1}^{\infty} 4\left(\frac{5^n}{3^{n+1}}\right)$ 17. $\sum_{n=3}^{\infty} \frac{(n-2)3^n}{n!}$

18. Which of the following series is/are equivalent to $\sum_{n=1}^{\infty} \frac{2n}{n+1}$?

a. $\sum_{n=0}^{\infty} \frac{2(n+1)}{n+2}$ b. $\sum_{n=0}^{\infty} \frac{2n}{n+1}$ c. $1+\sum_{n=2}^{\infty} \frac{2n}{n+1}$ d. $\sum_{n=1}^{\infty} \left(2-\frac{2}{n+1}\right)$ e. $\frac{7}{3}+\sum_{n=3}^{\infty} \frac{2n}{n+1}$

Selected Answers:

1. ∞, div. 2. $\frac{2}{3}$, conv. 3. 3, div. 4. 0, conv. 5. ∞, div. 6. 1, inconclusive

7. 2, div. 8. conv AST 10. conv. GST 11. conv. *p*ST 14. div. *n*TT

16. div. GST 17. conv. RT