LESSON 9-5 INTEGRATION INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Since $\frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}}$ and $\frac{d}{dx}\arctan u = \frac{u'}{1+u^2}$, it follows that

$$\boxed{\int \frac{u'}{\sqrt{1-u^2}}\,dx = \arcsin u + C \quad \text{and} \quad \int \frac{u'}{1+u^2}\,dx = \arctan u + C \quad (\text{where } u \text{ is a function of } x).}$$

Extending these integration rules gives us these more general **integration rules**.

Identify form
Find
a, u, u'

1. $\int \frac{u'}{\sqrt{a^2-u^2}}\,dx = \arcsin\frac{u}{a} + C$ 2. $\int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a}\arctan\frac{u}{a} + C$

Note: Since $\frac{d}{dx}\arcsin x$ and $\frac{d}{dx}\arccos x$ differ only in sign, it is not necessary to have a third integration rule which integrates into $\arccos x$.

Warm-up Example: Differentiate $y = \arcsin\frac{x}{2}$.

$y = \arcsin(\frac{1}{2}x)$

$y' = \frac{\frac{1}{2}}{\sqrt{1-(\frac{x}{2})^2}} = \frac{1}{2\sqrt{1-\frac{x^2}{4}}}$

$= \frac{1}{\sqrt{4-x^2}}$ $a=2$, $u=x$

Examples: Integrate.

arcsin form
$a=2$
$u=x$
$u'=1$

1. $\int \frac{1}{\sqrt{4-x^2}}\,dx$

$= \arcsin\frac{x}{2} + C$

arctan form
$a=5$
$u=2x$
$u'=2$

2. $\int \frac{dx}{4x^2+25} = \frac{1}{2}\int \frac{2}{25+4x^2}\,dx$

$= \frac{1}{2}\cdot\frac{1}{5}\arctan\frac{2x}{5} + C$

arctan form
$a=\sqrt{3}$
$u=2x$
$u'=2$

3. $\int \frac{8}{3+4x^2}\,dx = 4\int \frac{2}{3+4x^2}\,dx$

$= \frac{4}{\sqrt{3}}\arctan\frac{2x}{\sqrt{3}} + C$

4. $\int \frac{\overset{u'}{8x}}{\underset{u}{3+4x^2}}\,dx = \ln(3+4x^2) + C$

$\frac{u'}{u}$ form Easy if you see it!

Divide

5. $\int \frac{8x^2}{3+4x^2}\,dx = \int 2\,dx - 3\int \frac{2}{3+4x^2}\,dx$

$= 2x - 3\cdot\frac{1}{\sqrt{3}}\arctan\left(\frac{2x}{\sqrt{3}}\right) + C$

(See Example 3)

$$\begin{array}{r} 2 - \frac{6}{4x^2+3} \\ 4x^2+3 \overline{\big)\, 8x^2 \qquad} \\ \underline{-\,8x^2 + 6} \\ -6 \end{array}$$

6. $\int \frac{x+4}{\sqrt{4-x^2}}\,dx = \int \frac{x}{\sqrt{4-x^2}}\,dx + 4\int \frac{1}{\sqrt{4-x^2}}\,dx$

$= -\frac{1}{2}\int (4-x^2)^{-\frac{1}{2}}(-2x)\,dx + 4\int \frac{1}{\sqrt{4-x^2}}\,dx$

$= -(4-x^2)^{\frac{1}{2}} + 4\arcsin\frac{x}{2} + C$

(See Example 1)

<u>Examples 7.</u> Complete the square to find $\int \frac{1}{x^2+4x+8}\,dx$.

$$\int \frac{1}{x^2+4x+4+4}\,dx$$

$$\int \frac{1}{(x+2)^2+4}$$

$$\frac{1}{2}\arctan \frac{x+2}{2} + C$$