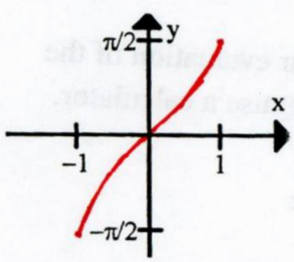
LESSON 9-4 INVERSE TRIGONOMETRIC FUNCTIONS, DIFFERENTIATING INVERSE TRIGONOMETRIC FUNCTIONS

None of the six basic trig functions is one-to-one, so none of them have an inverse function. However, we can use domain restrictions to make the trig functions one-to-one, so that they do have inverses. In this course, we will deal only with the inverse trig functions for the sine, cosine, and tangent functions.

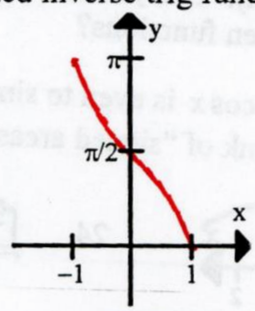
Definition of the Inverse Trig Functions:

Function	Domain (x values)	*Range (y values)
$y = \arcsin x \leftrightarrow \sin y = x$	$[-1,1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \arccos x \leftrightarrow \cos y = x$	$[-1,1]$	$[0, \pi]$
$y = \arctan x \leftrightarrow \tan y = x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

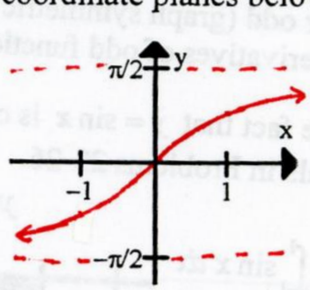
Example 1: Graph the indicated inverse trig functions in the coordinate planes below:



$y = \arcsin x$

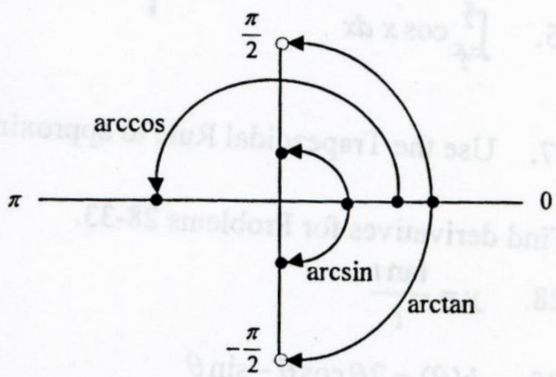


$y = \arccos x$



$y = \arctan x$

*A geometric representation of the range values for each inverse trig function is shown in the coordinate plane at right.



The two biggest mistakes that students make with inverse trig functions are: 1) putting an inappropriate range value for an answer or 2) putting more than one range value for an answer.

For example, $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$. That is the one and only correct answer (appropriate range value). $\arcsin\left(-\frac{1}{2}\right) \neq \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

Example 2: Explain why $\arcsin\left(-\frac{1}{2}\right) = \frac{11\pi}{6}$ and $\arcsin\left(-\frac{1}{2}\right) = \frac{7\pi}{6}$ are not true, even though $\sin\frac{7\pi}{6} = \sin\frac{11\pi}{6} = -\frac{1}{2}$.

Example 3: Evaluate without a calculator.

a. $\arctan 1 = \frac{\pi}{4}$ b. $\arccos(-1) = \pi$ c. $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

d. $\arcsin 2$ nonexistent

Example 4: Use a calculator to evaluate.

a. $\arcsin(0.3) = .304$ (or .305) b. $\arctan\left(-\frac{5}{2}\right) = -1.190$

Since (for appropriate values) $f(x) = \sin x$ and $g(x) = \arcsin x$ are inverse functions, $f\left(g(x)\right) = \sin(\arcsin x) = x$. Similarly, $\cos(\arccos x) = x$ and $\tan(\arctan x) = x$.

Also, $\arcsin(\sin x) = x$, but only when $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\arccos(\cos x) = x$, but only when $0 \le x \le \pi$

and $\arctan(\tan x) = x$ but only when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Example 5: Simplify without a calculator.

a. $\sin\left(\arcsin\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$ b. $\tan(\arctan 3) = 3$ c. $\arccos\left(\cos\frac{\pi}{3}\right) = \frac{\pi}{3}$

* d. $\arcsin\left(\sin\frac{11\pi}{6}\right) = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

Hint: Be careful on this one!

Example 6: Solve for x.

$\arcsin(x^2 - 3) = \frac{\pi}{2}$

$\sin\left(\arcsin\left(x^2-3\right)\right) = \sin\frac{\pi}{2}$

$x^2 - 3 = 1$

$x^2 = 4$

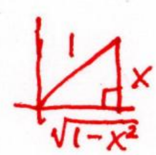
$x = \pm 2$

Examples: Sketch a right triangle, and evaluate without a calculator.

7. Find $\tan x$, given that $x = \arccos\frac{2}{\sqrt{5}}$

$\tan x = \frac{1}{2}$

8. Find $\cos y$, given that $y = \arcsin x$



$\cos y = \sqrt{1-x^2}$

Example 9: Use your work from Example 8 to find $\frac{d}{dx}\arcsin x$.

Hint: You will need to differentiate implicitly.

$y = \arcsin x \longrightarrow \sin y = x$

$\cos y \cdot y' = 1$

$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

Derivatives of the Inverse Trig Functions:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos u = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\arctan u = \frac{u'}{1+u^2}$$

(where u is a function of x)

Examples: Differentiate.

10. $g(y) = \arctan(2y-1)$

$g'(y) = \dfrac{2}{1+(2y-1)^2}$

11. $f(x) = \arcsin\sqrt{x}$ $= \arcsin(x^{\frac{1}{2}})$

$f'(x) = \dfrac{\frac{1}{2}x^{-\frac{1}{2}}}{\sqrt{1-(x^{\frac{1}{2}})^2}}$ or $\dfrac{1}{2\sqrt{x}\sqrt{1-x}}$

12. $h(t) = \arccos(\ln t)$

$h'(t) = \dfrac{-\frac{1}{t}}{\sqrt{1-(\ln t)^2}}$ or $\dfrac{-1}{t\sqrt{1-(\ln t)^2}}$

ASSIGNMENT 9-4

Evaluate the expressions in Problems 1-4 without using a calculator.

1. $\arcsin\frac{\sqrt{3}}{2}$ 2. $\arctan(-1)$ 3. $\arccos\left(\frac{-1}{2}\right)$ 4. $\arctan\sqrt{3}$

Use a calculator to evaluate in Problems 5-8.

5. $\arctan(-3)$ 6. $\arccos(.8)$ 7. $\arcsin\left(\frac{-1}{3}\right)$ 8. $\arccos\left(\sqrt{2}-1\right)$

Simplify the expressions in Problems 9-12 without using a calculator.

9. $\cos\left(\arccos\left(\frac{-2}{3}\right)\right)$ 10. $\tan\left(\arctan(2x+3)\right)$

11. $\arcsin\left(\cos\frac{\pi}{2}\right)$ 12. $\arctan\left(\tan\frac{4\pi}{3}\right)$ Be careful!

In Problems 13 and 14, solve for x without using a calculator.

13. $\arctan(3-x) = \frac{-\pi}{4}$ 14. $\arccos\left(x^2-2\right) = \pi$