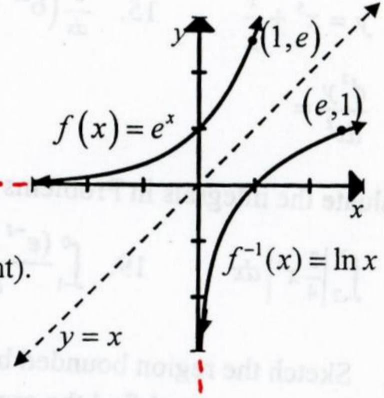
LESSON 7-4 LOGARITHMIC FUNCTIONS



Since $f(x)=e^x$ is one-to-one (continuous and increasing), it must have an inverse. However, if you switch x and y in the equation $y=e^x$ to get $x=e^y$, you cannot isolate the new y by using algebraic methods. So, we must define $f^{-1}(x)$ for the function $f(x)=e^x$. For $f(x)=e^x$, $f^{-1}(x)$ is called the **natural logarithmic function**, and we write $f^{-1}(x)=\ln x$ (so that $x=e^y$ and $y=\ln x$ must be equivalent).
In general, if $f(x)=a^x$ $(a>0)$, then $f^{-1}(x)=\log_a x$ (so that $x=a^y$ and $y=\log_a x$ must be equivalent).

Note: $\log_e x$ is usually written as $\ln x$ and $\log_{10} x$ is usually written simply as $\log x$.

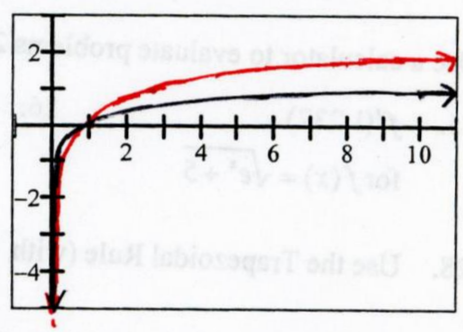
Graphs of Logarithmic Functions: If $f(x)=\log_a x$ and $a>1$, then

1. The domain of $f(x)$ is $(0,\infty)$. The range of $f(x)$ is $(-\infty,\infty)$.
2. The graph of $f(x)$ is continuous, increasing, concave downward, and one-to-one (has an inverse function).
3. The y-axis is a vertical asymptote downward: $\lim\limits_{x\to 0} f(x)=-\infty$ (Also, $\lim\limits_{x\to\infty} f(x)=\infty$)
4. The x-intercept is (1, 0). Another key point is $(a, 1)$.

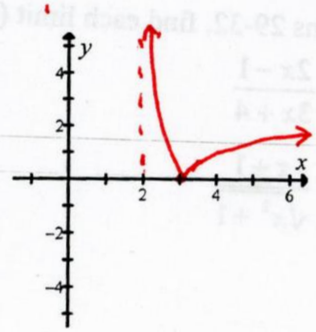
Compare these graphical characteristics of $f(x)=\log_a x$ to those of $f(x)=a^x$ from Lesson 5-1 (page 131).

Example 1: Use a calculator to graph $y=\ln x$ and $y=\log x$ in the same coordinate plane. Do you see any similarities in the graphs?

Satisfy characteristics in box above.



Example 2: Without using a calculator, sketch a graph of $y=|\ln(x-2)|$. Write an equation for the graph's asymptote.



For changing forms of an equation involving exponentials or logarithms, we use the following **Change of Form Definition**:

$$\text{Exponential form} \quad \left\{ \begin{array}{l} x = e^y \leftrightarrow y = \ln x \\ x = a^y \leftrightarrow y = \log_a x \end{array} \right\} \quad \text{Logarithmic form}$$

Example 3: Change the following equations from exponential form to logarithmic form or vice versa.

a. $3^4 = 81$ $\log_3 81 = 4$ b. $e^0 = 1$ $\ln 1 = 0$ c. $\log(.1) = -1$ $10^{-1} = .1$

Example 4:

a. Since $e^0 = 1$, $\ln 1 = 0$ b. Since $e^1 = e$, $\ln e = 1$

c. Because the natural exponential function and the natural logarithmic function are inverses, $\ln e^n = e^{\ln n} = n$

Example 5: Use the inverse idea from Example 4c. to simplify.

a. $\ln e^{\sqrt{2}} = \sqrt{2}$ b. $e^{\ln(3x)} = 3x$ c. $10^{\log 2} = 2$ d. $\log_2 2^{x^2} = x^2$

Properties of Logarithms:

1. $\ln(ab) = \ln a + \ln b$ These properties work for any bases,
2. $\ln \dfrac{a}{b} = \ln a - \ln b$ but only if $a > 0$ and $b > 0$
3. $\ln a^n = n \ln a$

Example 6: Expand using Logarithm Properties 1-3 above.

a. $\ln \frac{5}{8} = \ln 5 - \ln 8$

b. $\ln \sqrt[3]{x^2+1} = \ln(x^2+1)^{1/3}$
$= \frac{1}{3}\ln(x^2+1)$

Example 7: Condense into a single logarithm. ($x > 0$ and $y > 0$)

a. $-3\ln x + 5\ln y$
$\ln x^{-3} + \ln y^5$
$= \ln(x^{-3} \cdot y^5)$ or $\ln\left(\frac{y^5}{x^3}\right)$

b. $\frac{1}{2}\ln x + \ln(x+1) - 3\ln y$
$\ln x^{\frac{1}{2}} + \ln(x+1) - \ln y^3$
$= \ln \dfrac{x^{\frac{1}{2}}(x+1)}{y^3}$

Example 8: Solve for x.

a. $y = e^{2x-5} + 6$
$y - 6 = e^{2x-5}$
$\ln(y-6) = 2x - 5$
$5 + \ln(y-6) = 2x$
$\dfrac{5 + \ln(y-6)}{2} = x$

b. $\log_2 x - \log_2(x-8) = 3$
$\log_2 \dfrac{x}{x-8} = 3$
$2^3 = \dfrac{x}{x-8}$
$8 = \dfrac{x}{x-8}$
$8x - 64 = x$
$7x = 64$
$x = \dfrac{64}{7}$
Checks out

Change of Base Formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$

Since the only two logarithmic bases on your calculator are 10 (log key) and e (ln key), you will change bases on your calculator in one of two ways:

$$\log_a x = \frac{\log x}{\log a} \qquad \text{or} \qquad \log_a x = \frac{\ln x}{\ln a}$$

Example 9: Use your calculator to find $\log_7 112$ to 3 or more decimal places. $\log_7 112 = \dfrac{\log 112}{\log 7}$ or $\dfrac{\ln 112}{\ln 7} = 2.424$

Example 10:

a. Find an exact value for x, if $3^{x+2} = 6$. $\log_3 6 = x+2$

$x = -2 + \log_3 6$

b. Use your calculator to find a decimal value for your answer from Part a. to 3 or more decimal places. $x = -.369$

ASSIGNMENT 7-4

Decide whether each statement in Problems 1-8 is true or false for $a > 0$ and $b > 0$. (Check your answers before working on the rest of the assignment.)

1. $\log(a+b) = \log a + \log b$
2. $\ln(a+b) = \ln a \cdot \ln b$
3. $\log a - \log b = \dfrac{\log a}{\log b}$
4. $\log \dfrac{a}{b} = \dfrac{\log a}{\log b}$
5. $(\ln x)^3 = 3\ln x$
6. $\ln x^3 = 3\ln x$
7. $\ln x^2 = 2\ln x$, for all x
8. $\ln x^2 = 2\ln x$, for $x > 0$

For Problems 9-12, change each equation from exponential form to logarithmic form or vice versa.

9. $5^{-3} = \frac{1}{125}$
10. $e^x = 17$
11. $\log_3 729 = 6$
12. $\log x = -2$

Simplify each expression in Problems 13-16.

13. $e^{\ln(2x+1)}$
14. $\ln e^{a+b}$
15. $\log_5 5^{\sqrt{p}}$
16. $3^{\log_3 m^2}$

For Problems 17-20, solve for x without using a calculator. Simplify your answers.

17. $\log_2 x = 3$
18. $\ln x = -1$
19. $x^2 - 1 = \log_3 27$
20. $\log_x 64 = 3$