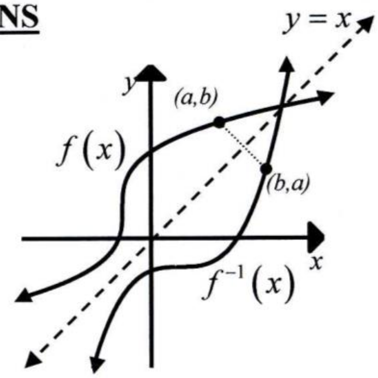
## LESSON 7-3 DERIVATIVES OF INVERSE FUNCTIONS

At right are the graphs of a function $f(x)$ and its inverse $f^{-1}(x)$. Remember that if the graph of $f$ contains the point $(a, b)$, then the graph of $f^{-1}$ contains the point $(b, a)$. Also, the graph of $f^{-1}$ is the reflection of the graph of $f$ across the line $y = x$.

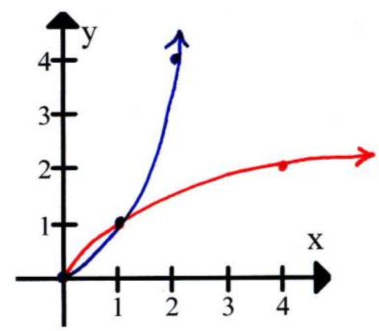



From the graphs above, do you see a relationship between the slope of the graph of $f$ at $(a, b)$ and the slope of the graph of $f^{-1}$ at $(b, a)$?

They are reciprocals.

Example 1: Let $f(x) = \sqrt{x}$.

a. Sketch the graph of $f(x)$.




b. Find $f^{-1}(x)$. Hint: You must list a domain restriction.

$x = \sqrt{y}$

$x^2 = y$ $\qquad f^{-1}(x) = x^2, \; x \geq 0$

c. Sketch the graph of $f^{-1}(x)$ in the same coordinate plane as the graph of $f(x)$.

d. Differentiate both $f(x)$ and $f^{-1}(x)$.

$f(x) = x^{\frac{1}{2}}$ $\qquad f^{-1}(x) = x^2$

$f'(x = \frac{1}{2}x^{-\frac{1}{2}}$ $\qquad (f^{-1})'(x) = 2x$

$= \frac{1}{2\sqrt{x}}$

e. Find the slope of the graph of $f(x)$ at (4, 2) and the slope of the graph of $f^{-1}(x)$ at (2, 4).

$f'(4) = \frac{1}{2\sqrt{4}}$ $\qquad (f^{-1})'(2) = 2 \cdot 2$

$= \frac{1}{4}$ $\qquad = 4$

f. What conclusion can you make about these slopes?

They are reciprocals.

Since slope $= m = \dfrac{\Delta y}{\Delta x}$, it should make sense that switching $x$ and $y$ (for inverse functions) should produce reciprocal slopes for inverse functions.

**Derivatives of Inverse Functions**:

If $(a, b)$ is a point on $f$, then $(b, a)$ is a point on $f^{-1}$, and $(f^{-1})'(b) = \dfrac{1}{f'(a)}$

or if $f$ and $g$ are inverse functions, then $g'(x) = \dfrac{1}{f'(g(x))}$.

Derivatives of inverses have reciprocal slopes at "image points" (points reflected across $y = x$). $(a, b)$ and $(b, a)$ are image points.

<u>Note</u>: When finding derivatives of inverse functions, do <u>not</u> use the same $x$-value for both $f$ and $f^{-1}$. This hardly ever works. (It only works when the $x$- and $y$-values of the ordered pairs are the same.)

<u>Example 2</u>: Let $f$ and $g$ be inverse functions such that:

| | | |
|---|---|---|
| $f(-1)=1$ | $f(0)=2$ | $f(1)=5$ |
| $f'(-1)=\frac{3}{2}$ | $f'(0)=2$ | $f'(1)=\frac{1}{2}$ |

Build a table

| $f'$ | $f$ | $g=f^{-1}$ | $g'$ |
|---|---|---|---|
| $f'(-1)=\frac{3}{2}$ | $(-1,1)$ | $(1,-1)$ | $g'(1)=\frac{2}{3}$ |
| $f'(0)=2$ | $(0,2)$ | $(2,0)$ | $g'(2)=\frac{1}{2}$ |
| $f'(1)=\frac{1}{2}$ | $(1,5)$ | $(5,1)$ | $g'(5)=2$ |

From the given information, find each of the following if possible.
<u>Hint</u>: Make a table or chart to organize your data.

a. $g'(1) = \frac{2}{3}$ b. $g'(2) = \frac{1}{2}$ c. $g'(3)$ can't determine d. $g'(0)$ can't determine e. $g'(5) = 2$

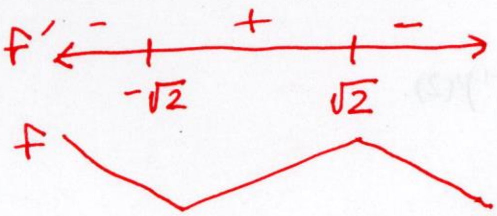
If a function $f$ has an inverse function $f^{-1}$, then $f$ is one-to-one and must be either strictly increasing or strictly decreasing (strictly monotonic) on its entire domain. We can use $f'$ to find out where $f$ is increasing and where $f$ is decreasing.

<u>Example 3</u>:

a. Use $f'(x)$ to show that $f(x) = 6x - x^3$ is not one-to-one on its entire domain.

$f'(x) = 6 - 3x^2 = 3(2 - x^2)$ $f$ is not one-to-one

$f' = 0$ @ $x = \pm\sqrt{2}$ (C.N.)




b. Find the largest interval containing $x = 0$ for which $f$ is one-to-one. $[-\sqrt{2}, \sqrt{2}]$

c. Find the largest interval containing $x = -2$ for which $f$ has an inverse function. $(-\infty, -\sqrt{2}]$