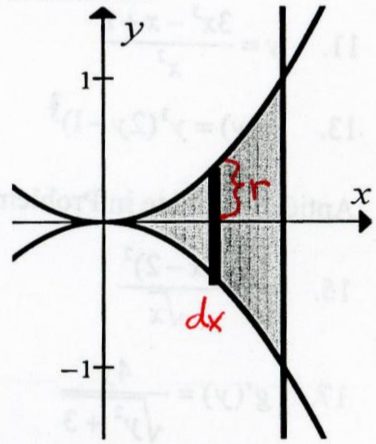
LESSON 6-6 VOLUMES OF SOLIDS OF REVOLUTION: DISCS AND WASHERS

If, for a given base, you create semicircular cross sections "sticking toward you" and "away from you," you have created circular cross sections. Since the formula for the area of a circle is $A = \pi r^2$, the formula for a volume with circular cross sections is:

$V = \int_a^b \pi r^2\,dx$ or $V = \pi \int_a^b r^2\,dx$ perpendicular to x-axis

$V = \pi \int_a^b r^2\,dy$ perpendicular to y-axis



Warm-up Example: Use the region bounded by $y = x^2$, $y = -x^2$, $x = 0$ and $x = 1$ as shown in the graph at right to create a volume by using circular cross sections perpendicular to the x-axis.

$d = x^2 - (-x^2) = 2x^2$

$r = x^2$

$V = \int_0^1 \pi (x^2)^2\,dx$

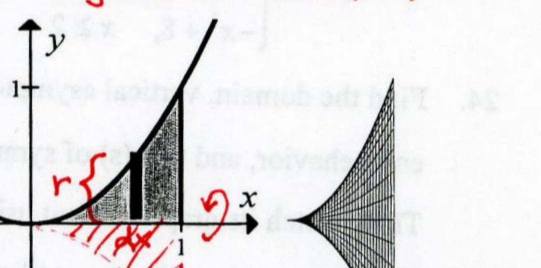
Circular cross sections can also be formed by revolving very thin (essentially no width) rectangles about an axis of revolution. These circular cross sections are more commonly called discs.

A volume formed by revolving a region about a line that does not pass through the interior of the region is called a solid of revolution. The line is called the axis of revolution.

If a region is bounded by the axis of revolution, the volume of the solid of revolution is a sum of the volumes of essentially an infinite number of cylindrical discs.

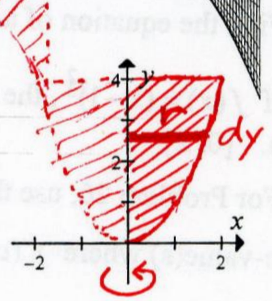
Disc Formula: $V = \pi \int_a^b r^2$ (dx or dy) r = top (of brace) − bottom (of brace) or right (o.b.) − left (o.b.)

Example 1: Set up an integral for the volume of the solid formed by revolving the region bounded by $y = x^2$, $y = 0$, and $x = 1$ about the x-axis.



$V = \pi \int_0^1 (x^2)^2\,dx$

Example 2: Find the volume of the solid formed by revolving the region in Quadrant I bounded by $y = x^2$, $x = 0$, and $y = 4$ about the y-axis.



$x = \sqrt{y}$ $V = \pi \int_0^4 (\sqrt{y})^2\,dy$

$= \pi \int_0^4 y\,dy = \pi \cdot \frac{1}{2} y^2 \Big|_0^4$

$= 8\pi - 0 = 8\pi$

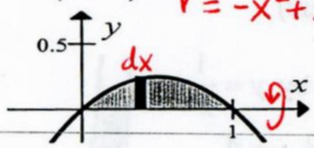
When a region is revolved about a line which is not one of its boundaries, its volume is formed from a sum of volumes of washers (at least some of the discs have holes in them).

Washer Formula:		$V = \pi \int_a^b (R^2 - r^2)\ (dx \text{ or } dy)$
R	=	Outer radius (from the axis of revolution)
r	=	Inner radius (from the axis of revolution)

Example 3: Set up integrals for the volumes of the solids formed by revolving the region bounded by $y = -x^2 + x$ and $y = 0$

a. about the x-axis (Discs)

$r = -x^2 + x$



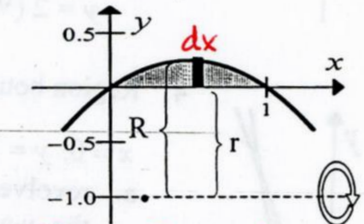
Limits: $-x^2 + x = 0$

$-x(x-1) = 0$

$x = 0, 1$

$V = \pi \int_0^1 (-x^2 + x)^2\,dx$

b. about $y = -1$ (Washers)

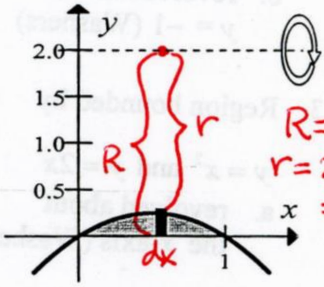


$R = -x^2 + x + 1$

$r = 1$

$V = \pi \int_0^1 ((-x^2 + x + 1)^2 - 1^2)\,dx$

c. about $y = 2$ (Washers)



$R = 2$

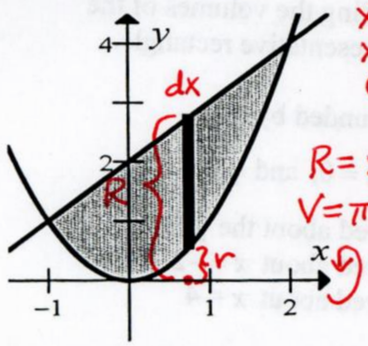
$r = 2 - (-x^2 + x)$

$= 2 + x^2 - x$

$V = \pi \int_0^1 (2^2 - (2 + x^2 - x)^2)\,dx$

Example 4: Set up integrals for the volumes of the solids formed by revolving the region bounded by $y = x^2$ and $y = x + 2$

a. about the x-axis



Limits:

$x^2 = x + 2$

$x^2 - x - 2 = 0$

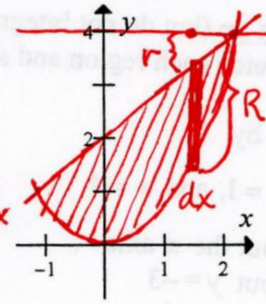
$(x-2)(x+1) = 0$

$x = 2, -1$

$R = x + 2 \quad r = x^2$

$V = \pi \int_{-1}^{2} ((x+2)^2 - (x^2)^2)\,dx$

b. about the line $y = 4$



$R = 4 - x^2$

$r = 4 - (x+2) = 2 - x$

$V = \pi \int_{-1}^{2} ((4 - x^2)^2 - (2 - x)^2)\,dx$

Note: When finding volumes using either discs or washers, you should always sketch your region and draw a representative rectangle. Your figure will help you decide whether to use dx or dy (the representative rectangles must be perpendicular to the axis of revolution). The figure can also help you choose a method (discs or washers), find limits of integration, and decide which expressions to use for R and r (when using washers). Your integral must be consistent ("all x" with dx or "all y" with dy).

Remember: Perpen**Disc**ular