LESSON 6-5 VOLUMES OF SOLIDS WITH KNOWN CROSS SECTIONS

In Geometry, you learned formulas for finding volumes of common three-dimensional solids (cubes, spheres, cones, rectangular prisms, and perhaps others).

Calculus allows us to find volumes of solids whose bases are two-dimensional regions within an *x-y* coordinate system, and whose heights are formed by cross sections (most often squares, rectangles, semicircles, or triangles) which essentially "stick out from the base" to form the third dimension of the object.

Top – Bottom or Right – Left = b or s or d

Area formulas for common cross sections:

Square	Rectangle	* Semicircle	Triangle	Equilateral Triangle
$A = s^2$	$A = bh$	$A = \frac{1}{2}\pi r^2$	$A = \frac{1}{2}bh$	$A = \frac{\sqrt{3}}{4}s^2$

$r = \frac{1}{2}d$

Volumes of Solids with Known Cross Sections

For cross sections perpendicular to the *x*-axis:

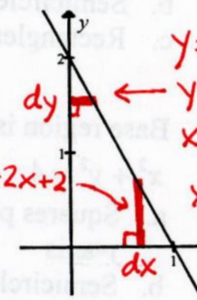
$V = \int_a^b A\,dx$, (dx = width) where A is a function of x and gives the area of a representative cross section.

For cross sections perpendicular to the *y*-axis:

$V = \int_a^b A\,dy$, (dy = width) where A is a function of y.

Examples:

1. Find the volume of the solid whose base is a triangle bounded by $y = -2x + 2$, $x = 0$, and $y = 0$, and whose cross sections are squares which are perpendicular to the *x*-axis.



$y = -2x + 2$
$y - 2 = -2x$
$x = \frac{y-2}{-2}$
$x = -\frac{1}{2}y + 1 = b$

$V = \int_a^b A\,dx$

$V = \int_0^1 s^2\,dx = -\frac{1}{2}\int_0^1 (-2x+2)^2 \, dx(-2) = -\frac{1}{2}\cdot\frac{1}{3}(-2x+2)^3\Big|_0^1$

$= -\frac{1}{6}(0^3) - -\frac{1}{6}(2^3) = \frac{8}{6} = \frac{4}{3}$

2. Set up (but do not integrate) integrals for the volumes of the solids with the same base as in Example 1, but whose cross sections are:

   a. Semicircles perpendicular to the *x*-axis. $r = \frac{1}{2}(-2x+2) = -x+1$

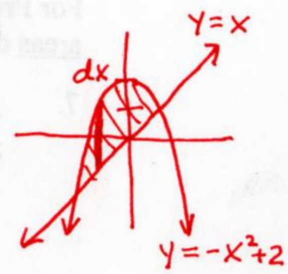
   $V = \frac{1}{2}\pi\int_0^1 (-x+1)^2\,dx$

   * b. Rectangles of height $\frac{1}{4}$ which are perpendicular to the *y*-axis.

   $V = \int_0^2 \left(-\frac{1}{2}y+1\right)\cdot\frac{1}{4}\,dy$

3. Set up (but do not integrate) an integral for the volume of a solid whose base is bounded by $y = -x^2 + 2$ and $y = x$ and whose cross sections are squares perpendicular to the *x*-axis.

$-x^2 + 2 = x$
$0 = x^2 + x - 2$
$0 = (x+2)(x-1)$
$x = -2, 1$

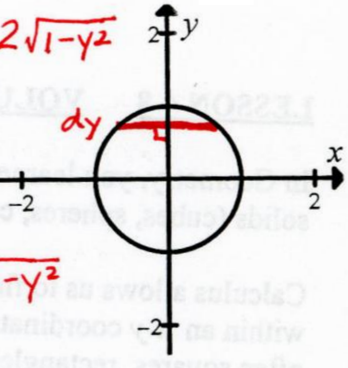


$s = \text{top} - \text{bottom} = -x^2 + 2 - x$

$V = \int_{-2}^{1} (-x^2 + 2 - x)^2\,dx$

Right − Left

$x^2 = 1 - y^2 \quad x = \pm\sqrt{1-y^2} \quad s = \sqrt{1-y^2} - (-\sqrt{1-y^2}) = 2\sqrt{1-y^2}$



4. Set up integrals for the volumes of the solids whose base is the circle $x^2 + y^2 = 1$ and whose cross sections are:

   a. Equilateral triangles perpendicular to the y-axis.

   $V = \frac{\sqrt{3}}{4}\int_{-1}^{1} (2\sqrt{1-y^2})^2\,dy$

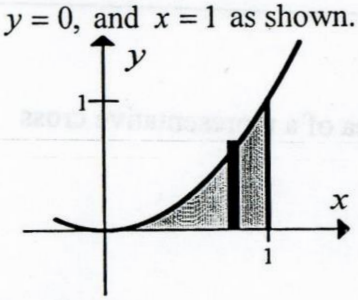
   b. Rectangles whose heights are three times their bases and whose bases are perpendicular to the y-axis. $b = 2\sqrt{1-y^2}$

   $V = \int_{-1}^{1} (2\sqrt{1-y^2})(3 \cdot 2\sqrt{1-y^2})\,dy$

ASSIGNMENT 6-5

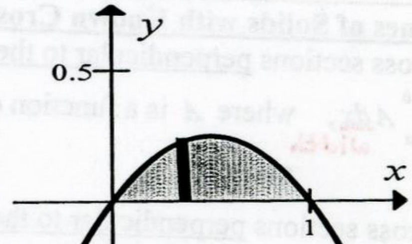
For Problems 1-4, set up (but do not integrate) integrals for evaluating the volumes of the solids formed by the given cross sections. For Problems 3-6, sketch the graphs of the regions before setting up the integrals used for finding the volumes.

1. Base region bounded by $y = x^2$, $y = 0$, and $x = 1$ as shown.



   a. Squares
   b. Semicircles
   c. Rectangles with height 2

2. Base region bounded by $y = -x^2 + x$ and $y = 0$ as shown.



   a. Squares
   b. Equilateral triangles
   c. Rectangles whose heights are half of their bases

3. Base region is the circle $x^2 + y^2 = 4$

   a. Squares perpendicular to the y-axis
   b. Semicircles perpendicular to the y-axis.

4. Base region bounded by $y = 2 - x$ and $y = x^2$

   a. Squares perpendicular to the x-axis
   b. Right triangles perpendicular to the x-axis. The base of the triangle sits in the region and the height $= 3$ times the base.

5. Find the volume of a solid whose base is bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$, and whose cross sections are squares perpendicular to the x-axis.

6. Find the volume of a solid whose base is bounded by $y = 2x$, $y = 0$, and $x = 3$, and whose cross sections are semicircles perpendicular to the x-axis.

For Problems 7 and 8, sketch the regions bounded by the given equations, and find the areas of the regions. Show your integral set ups, and do not use a calculator.

7. $f(x) = 2x^2 + 3x$
   $g(x) = 2$

8. $f(y) = y^2 - 2y$
   $g(y) = 2 - y$