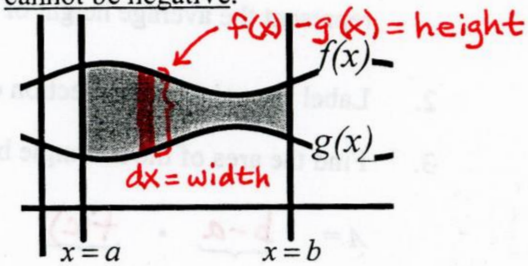
## LESSON 6-4 AREA BETWEEN CURVES, AVERAGE VALUE OF A FUNCTION

$\int_a^b f(x)\,dx$ produces a value ("signed area") which may be positive, negative, or zero.

However, if you are asked to find an actual area, that area cannot be negative.

**Area of a Region Between Two Curves**

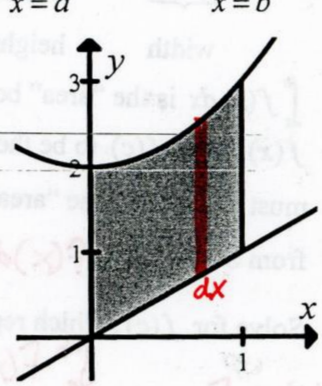
$A = \int_a^b \underbrace{(f(x) - g(x))}_{\text{height}}\,\underbrace{dx}_{\text{width}}$ if $f(x)$ and $g(x)$ are continuous and $f(x) \ge g(x)$ on $[a,b]$.




For functions of $x$, $A = \int_a^b (\text{top curve} - \text{bottom curve})\,dx$.

For functions of $y$, $A = \int_a^b (\text{right curve} - \text{left curve})\,dy$.

Example 1: Find the area of the region bounded by $y = x^2 + 2$, $y = x$, $x = 0$, and $x = 1$.



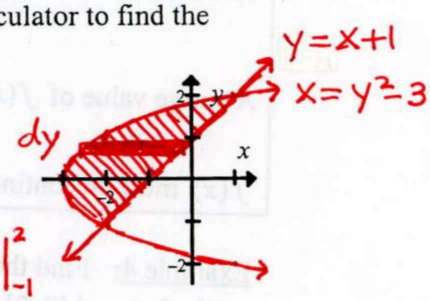

$\int_0^1 (x^2+2-x)\,dx = \left(\frac{1}{3}x^3 + 2x - \frac{1}{2}x^2\right)\Big|_0^1$

$= \left(\frac{1}{3}+2-\frac{1}{2}\right) - (0+0-0) = \frac{2}{6} + \frac{12}{6} - \frac{3}{6} = \frac{11}{6}$

Sometimes you have to find where two curves intersect to determine "boundaries" for your region(s). These intersections will provide you with limits of integration for your integral(s). You must show an equation set up, even when using a calculator to find the intersections.

Example 2: (Functions of $y$)

Find the area of the region bounded by $x = y^2 - 3$ and $y = x + 1$.




$x = y^2 - 3$ and $x = y - 1$

$y^2 - 3 = y - 1$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y = 2, -1$

$A = \int_{-1}^{2} (y - 1 - (y^2 - 3))\,dy$

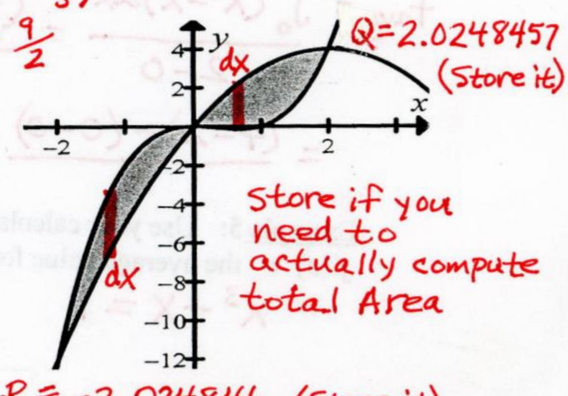
$= \int_{-1}^{2} (y + 2 - y^2)\,dy = \left(\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right)\Big|_{-1}^{2}$

$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$

$= 8 - \frac{9}{3} - \frac{1}{2} = 5 - \frac{1}{2} = \frac{9}{2}$

Example 3: Set up integrals for the total area of the regions located between the two curves as shown. You may use a calculator.

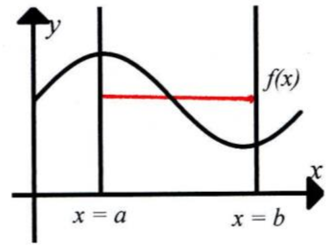
$f(x) = x^3 - x^2$ and $g(x) = -x^2 + 4.1x$




$x^3 - x^2 = -x^2 + 4.1x \longrightarrow x = -2.025, 0, 2.025$

$A = \int_P^0 (f(x) - g(x))\,dx + \int_0^Q (g(x) - f(x))\,dx$

$P = -2.024846$ (Store it)




Discovering the Formula for the Average Value of a Function.
The average value of a function represents its average "height."

1. Draw a horizontal segment from $x = a$ to $x = b$ in the figure at right which could represent the average height of $f(x)$ on $[a, b]$.

2. Find the area of the rectangle formed.

$A = \underbrace{(b-a)}_{\text{width}} \cdot \textit{height}$

The area of this rectangle is the same as the area under the curve.

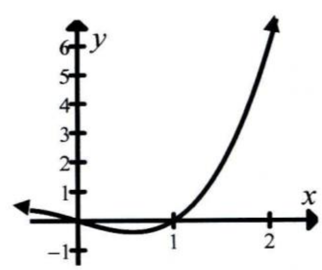
3. $\int_a^b f(x)\,dx = (b-a) \cdot \textit{height}$ Now solve for the average height.

$\dfrac{\int_0^b f(x)\,dx}{b-a} = \text{height}$

**Average Value of a Function:**

Average value of $f(x)$ on $[a, b]$ = $f(c)$ = $\boxed{f_{avg} = \dfrac{\overbrace{\int_a^b f(x)\,dx}^{\text{"area" of region}}}{\underbrace{b-a}_{\text{"width" of region}}}}$ or $\dfrac{1}{b-a}\int_a^b f(x)\,dx$

$f(x)$ must be continuous on $[a, b]$.




Example 4: Find the average value of $f(x) = x^3 - x$ on the interval [0, 2] without using a calculator.

$$f_{avg} = \frac{\int_0^2 (x^3 - x)\,dx}{2-0}$$
$$= \frac{\left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right)\Big|_0^2}{2}$$
$$= \frac{1}{2}\left((4-2)-(0-0)\right) = 1$$

Example 5: Use your calculator to find the value of $c$ in the interval [0, 2] where $f(c)$ = the average value found in Example 4.

$x^3 - x = 1$

$x = 1.324$ or $1.325$