LESSON 5-5 THE GENERAL POWER RULE FOR INTEGRALS AND *U*-SUBSTITUTION

In Lesson 3-3, you learned to differentiate composite functions by using the General Power Rule for Derivatives (Chain Rule for power functions). $\frac{d}{dx}u^n = nu^{n-1}u'$ (where u is a function of x). We reverse this process when integrating.

General Power Rule for Integrals: (Informally called the Reverse Chain Rule)

$$\int u^n u' dx = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

This looks a lot like the simple power rule for integration that you learned in the last lesson. However, the general power rule requires a "hook-on factor" u' to be present before you can integrate. It is a crucial part of the "Reverse Chain Rule."

Examples:

1. Differentiate $(1+5x)^4$ $\quad \frac{d}{dx}(1+5x)^4 = 4(1+5x)^3 \cdot 5$

2. Now, integrate $5(1+5x)^3$ $\quad \int 5(1+5x)^3\,dx = \frac{(1+5x)^4}{4} + C$

Note: You "hooked on" the derivative of the inside of the power function in Example 1, so you had to "unhook" the derivative of the inside in Example 2.

Examples: Integrate.

3. $\int (3x-1)^{10}\,dx$

$\frac{1}{3}\int 3(3x-1)^{10}\,dx$

$\frac{1}{3}\frac{(3x-1)^{11}}{11} + C$

4. $\int (3t^2+2t)(t^3+t^2)\,dt$

$\frac{(t^3+t^2)^2}{2} + C$

5. $\int \frac{6x^2}{\sqrt{4x^3-5}}\,dx$

$\frac{1}{2}\int (4x^3-5)^{-\frac{1}{2}} \cdot 2\cdot 6x^2\,dx$

$\frac{1}{2}\frac{(4x^3-5)^{\frac{1}{2}}}{\frac{1}{2}} + C$

$\sqrt{4x^3-5} + C$

6. $\int (y^3+1)^2\,dy$

Note: there is no "hook on". This must be expanded. "Reverse Chain Rule" does not work.

$\int (y^6+2y^3+1)\,dy$

$\frac{1}{7}y^7 + \frac{1}{2}y^4 + y + C$ Term by term integration

<u>*u*-Substitution</u>

For more complicated integration problems, simple rules for integration might fail, and you may have to make some type of substitution to be able to integrate. In this course, a common substitution will be to let $u =$ the radicand$\left(\sqrt{\text{radicand}}\right)$ part of the expression and to change the variable throughout the integral before integrating. You should use this method of substitution (called u-substitution) only when simpler methods don't work. It should be your last resort.

Procedure for u-substitution: (for $\int \_\_\_\_ \, dx$ problems requiring the method)

1. Let $u =$ radicand (part inside the $\sqrt{\ }$ symbol).
2. Solve for x (in terms of u).
3. Differentiate the equation from Step 2.
4. Find dx.
5. Substitute u-expressions for x-expressions in the integral.
   <u>Note</u>: Most often, $dx \neq du$. Don't forget to substitute for dx.
6. Integrate.
7. Substitute back, so that your final answer is again in terms of x.

Sometimes it is easier to do Step 3 before Step 2. These two steps are reversible.

<u>Examples</u>: Integrate.

7. $\int x\sqrt{x-1}\, dx$

   Why doesn't the "Reverse Chain Rule" work here?

Let $u = x-1$
$u+1 = x$
$du = dx$

$$\begin{aligned}\int x\sqrt{x-1}\,dx &= \int (u+1)\sqrt{u}\,du \\ &= \int (u+1)u^{\frac{1}{2}}\,du \\ &= \int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du \\ &= \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C\end{aligned}$$

8. $\int \frac{2x-1}{\sqrt{2x+3}}\, dx$

Let $u = 2x+3$
$u-3 = 2x$
$\frac{1}{2}u - \frac{3}{2} = x$
$\frac{1}{2}du = dx$

$$\begin{aligned}\int \frac{2x-1}{\sqrt{2x+3}}\,dx &= \int \frac{2\left(\frac{1}{2}u-\frac{3}{2}\right)-1}{u^{\frac{1}{2}}}\cdot\frac{1}{2}du \\ &= \int \frac{\frac{1}{2}u - \frac{3}{2} - \frac{1}{2}}{u^{\frac{1}{2}}}\,du \\ &= \int \frac{\frac{1}{2}u - 2}{u^{\frac{1}{2}}}\,du \\ &= \int \left(\frac{1}{2}u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}\right)du \\ &= \frac{1}{3}u^{\frac{3}{2}} - 4u^{\frac{1}{2}} + C \\ &= \frac{1}{3}(2x+3)^{\frac{3}{2}} - 4(2x+3)^{\frac{1}{2}} + C \\ &= \frac{1}{3}\left(\sqrt{2x+3}\right)^{3} - 4\sqrt{2x+3} + C\end{aligned}$$

You now have three strategies for integrating.

1. Term by term using the rules from page 90.
2. General Power Rule (Reverse Chain Rule).
3. u-substitution.