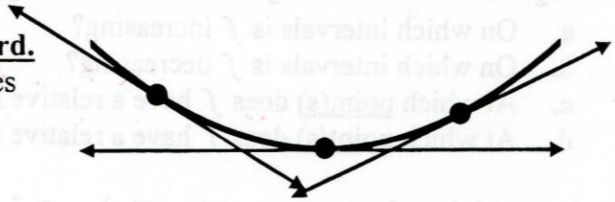
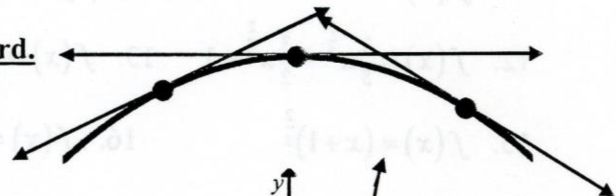
LESSON 4-3 CONCAVITY AND POINTS OF INFLECTION, THE SECOND DERIVATIVE TEST FOR RELATIVE EXTREMA

A graph with this shape is called **concave upward.** The tangent lines lie **below** the graph. The slopes of the tangent lines are increasing which means $f''(x) \geq 0$.

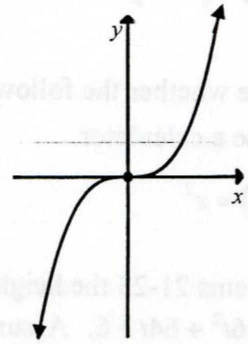


A graph with this shape is called **concave downward.** The tangent lines lie **above** the graph. The slopes of the tangent lines are decreasing which means $f''(x) \leq 0$.



Nonmathematical Memory Device:
Concave upward ↔ positive ↔ smiley face ↔ 

Concave downward ↔ negative ↔ frowny face ↔ 

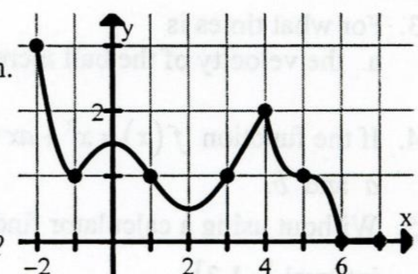


A point on a graph is a **point of inflection** if:

1. the graph has a tangent line at that point **and**
2. the graph changes concavity at that point.

On the graph of $y = x^3$ shown, the point (0,0) is a point of inflection.

Examples: Use the graph at the right to answer these questions. Base your answers on appearances of the graph.



1. On which intervals is the graph concave upward?
   (-2,-1), (1,3), (4,5)
2. On which intervals is the graph concave downward?
   (-1,1), (5,6)
3. On which intervals does the graph have no concavity?
   (3,4), (6,7)
4. What are the points of inflection?
   (1,1), (5,1)

Analytically we find concavity intervals and points of inflection by using a second derivative number line.
The procedure is parallel to the procedure used in the last lesson to find increasing/decreasing intervals and relative extrema by using a first derivative number line.

Examples:

5. Determine the points of inflection and discuss the concavity for the graph of

$f(x) = x^4 + x^3 - 3x^2 + 1$.

D: Reals

$f'(x) = 4x^3 + 3x^2 - 6x$

$f''(x) = 12x^2 + 6x - 6$

$= 6(2x^2 + x - 1)$

$= 6(2x-1)(x+1)$

f'' is never undefined

$f'' = 0$ when $x = \frac{1}{2}$ or $x = -1$

P.P.I: $x = -1, \frac{1}{2}$

f'': $+$ | -1 | $-$ | $\frac{1}{2}$ | $+$

concave up: $(-\infty, -1), (\frac{1}{2}, \infty)$

concave down: $(-1, \frac{1}{2})$

P.I: $(-1, -2)$

$(\frac{1}{2}, \frac{7}{16})$

6. If $f(x) = \dfrac{x^2+1}{x^2-4}$ and $f''(x) = \dfrac{10(3x^2+4)}{(x^2-4)^3}$, list the intervals where the graph of f is concave upward, concave downward, and list the points of inflection.

D: $x \neq \pm 2$

$f''(x) \neq 0$

f'' is undefined when $x = \pm 2$

(NOT PPI)

f'': $+$ | -2 | $-$ | 2 | $+$

concave up: $(-\infty, -2), (2, \infty)$

concave down: $(-2, 2)$

no P.I.

THE SECOND DERIVATIVE TEST FOR RELATIVE EXTREMA

This test does not require a second derivative number line. It does not find points of inflection. It is used to find relative extrema (max/min).

Procedure:

1. Use f' to find critical numbers.
2. Plug critical numbers into f'' and analyze concavity to determine if the function has a relative minimum or maximum.

Note: The Second Derivative Test does not always give an answer (when $f''(x) = 0$). Use it only when the directions require it or when the given information requires it.

Examples:

7. Use the Second Derivative Test to find the relative minimum and relative maximum points for the graph of $f(x) = -3x^4 + 6x^2$.

$f'(x) = -12x^3 + 12x$

$= -12x(x^2 - 1)$

$= -12x(x+1)(x-1)$

$f'(x) = 0$ when $x = 0, \pm 1$

CN: $x = 0, \pm 1$

$f''(x) = -36x^2 + 12$

$f''(0) = 12$ $+$ Rel. min. at $x = 0$

$f''(-1) = -24$ $-$ Rel. max. at $x = -1$

$f''(1) = -24$ $-$ Rel. max. at $x = 1$

Points: Rel. min: $(0, 0)$

Rel. max: $(\pm 1, 3)$

8. $g(x)$ is a function such that $g'(-3)=0,\ g'(-1)=0,\ g'(0)=0,$ and $g'(2)=0$.
If $g''(-3)=4,\ g''(-1)=-2,\ g''(0)=0,\ g''(1)=5,$ and $g''(2)=3$, find the x-values of the relative maximum and relative minimum values when possible.

C.N: $x=-3,-1,0,2$

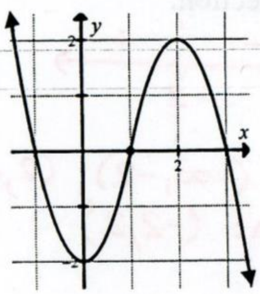
g'' + − N.A. +

Rel. max. at $x=-1$
Rel. min. at $x=-3, 2$

ASSIGNMENT 3-3

Use the appearance of these graphs to answer these three questions.

a. On which interval(s) is the graph of the function concave upward?
b. On which interval(s) is the graph of the function concave downward?
c. What are the points of inflection?

1.


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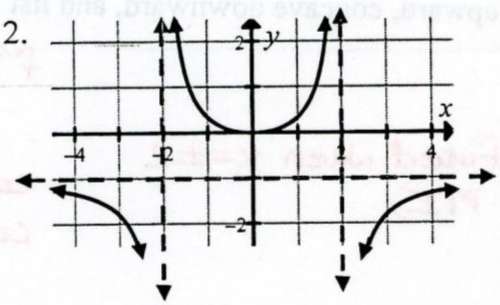

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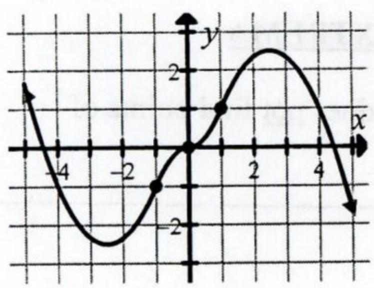
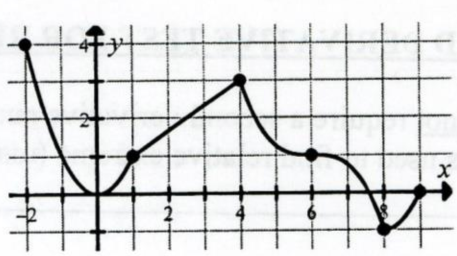


4.


Show organized steps and an f'' number line to answer the same three questions for these functions.

5. $f(x)=x^4-4x^3+2$ 6. $f(x)=3x^5-5x^4$ 7. $f(x)=x^{\frac{2}{3}}-3$

Use the Second Derivative Test to find the relative extrema points (see Example 7 on the previous page).

8. $f(x)=x^3-3x^2+6$ 9. $f(x)=-\frac{1}{4}x^4+\frac{9}{2}x^2+5$

For these problems, find the x-values of relative minimum points and the x-values of relative maximum points.

10. -3, 1, and 3 are critical numbers of f and $f''(-3)=-2,\ f''(1)=0$, and $f''(3)=2$.

11. $f'(-2)=f'(0)=f'(4)=0$

f'' + | − | +
at -1 and 2