# LESSON 4-1 ABSOLUTE EXTREMA AND THE MEAN VALUE THEOREM

Definitions (informal)

- The **absolute maximum** (global maximum) of a function is the $y$-value that is greater than or equal to all other $y$-values in the function.
- The **absolute minimum** (global minimum) of a function is the $y$-value
- that is less than or equal to all other $y$-values in the function.
- A **relative maximum** (local maximum) of a function is a $y$-value that is greater than or equal to all "nearby" $y$-values in the function.
- A **relative minimum** (local minimum) of a function is a $y$-value that is
- less than or equal to all "nearby" $y$-values in the function.
- **Extrema** (extreme values) are either maximum values (maxima) or minimum values (minima).
- **Critical Numbers** are $x$-values at which $f(x)$ exists but $f'(x)$ is either zero or undefined.

**PROCEDURE FOR FINDING ABSOLUTE (GLOBAL) EXTREMA:**

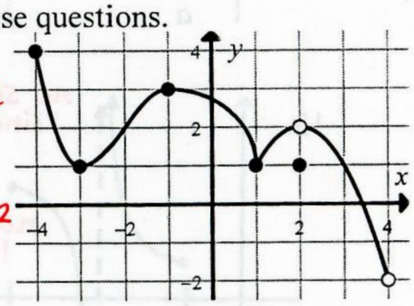
1. Find all critical numbers of the function.
2. Find $y$-values at each critical number **and** at each endpoint of the interval.
3. Choose the least and greatest $y$-values as absolute extrema.

**Note:**

Absolute extrema can occur either at critical numbers or endpoints.
Relative extrema can occur only at critical numbers. We will not consider endpoint extrema to be relative extrema.

Examples: Use the figure of $y = f(x)$ at the right to answer these questions.

1. What is the absolute maximum of $f$? 4
2. At what $x$-value does $f$ have an absolute maximum? $x = -4$
3. What is the absolute maximum point on $f$? $(-4, 4)$
4. What is the absolute minimum of $f$? There is none.
5. At what $x$-value(s) does $f$ have a relative minimum? $x = -3, 1, 2$
6. At what $x$-value(s) does $f$ have a relative maximum? $x = -1$




Examples:

7. Find the global extrema of $f(x) = \frac{1}{3}x^3 - 2x^2$ on the interval $[-1, 3]$.

$f'(x) = x^2 - 4x$
$\quad = x(x-4)$

$f'(x) = 0$ when $x = 0, 4$
but $x = 4$ is not in the interval
$f'(x)$ is never undefined
C.N: $x = 0$

$f(-1) = -\frac{1}{3} - 2 = -2\frac{1}{3}$ or $-\frac{7}{3}$
$f(0) = 0$
$f(3) = 9 - 18 = -9$

$\left.\begin{array}{l}\text{max. } f = 0 \\ \text{min. } f = -9\end{array}\right\}$ on $[-1, 3]$

8. Find the absolute maximum and absolute minimum values of $f(x)=|x-2|$ on the interval $[0,5]$.

$f(0)=2$ $\quad$ max. $f=3$

$f(2)=0$ $\quad$ min. $f=0$

$f(5)=3$

C.N: $x=2$
(sharp turn)
$f'(2)$ is undefined

9. Find the extrema of $f(x)=3x^{\frac{2}{3}}-2x$ on $[-1,3]$.

$f'(x)=2x^{-\frac{1}{3}}-2=\frac{2}{x^{\frac{1}{3}}}-2$ $\quad$ $f'(x)$ is undefined when $x=0$

$\frac{2}{x^{\frac{1}{3}}}-2=0$ $\quad$ $2=2\sqrt[3]{x}$ $\quad$ C.N: $x=0,1$ $\quad$ $f(-1)=5$ $\quad$ max. $f=5$

$\frac{2}{\sqrt[3]{x}}=2$ $\quad$ $1=\sqrt[3]{x}$ $\quad$ $f(0)=0$ $\quad$ min. $f=0$

$1=x$ $\quad$ $f(1)=1$

$f(3)=3\sqrt[3]{9}-6$

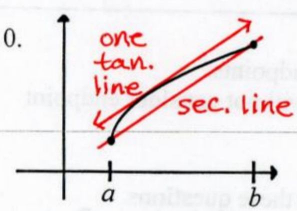
Discovering the Mean Value Theorem

For Examples 10-16 draw these lines (if possible).

(a) Draw the secant line between the two points $(a,f(a))$ and $(b,f(b))$.

(b) Draw all tangent lines parallel to the secant line.

10.

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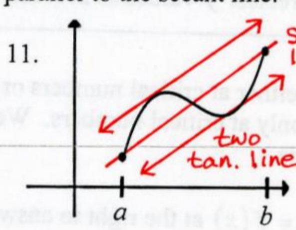


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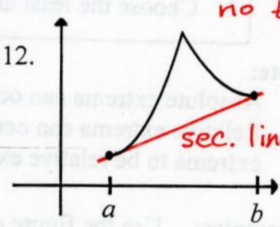


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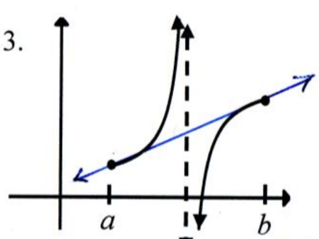
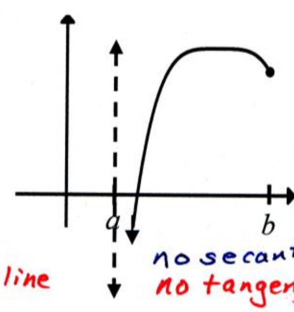


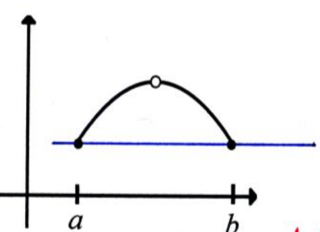

not really a secant line
no tangent line

14.




no secant line
no tangent line

15.




no tangent line

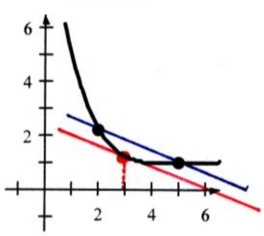
For examples 16-18

(a) Draw the secant line between the two points $(2,f(2))$ and $(5,f(5))$.

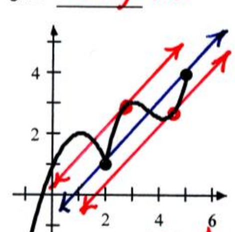
(b) Draw all tangent lines parallel to the secant line at some point on the interval (2,5).

(c) Estimate the value of $c$ where $(c,f(c))$ is a point of tangency.

16. $c\approx$ 3

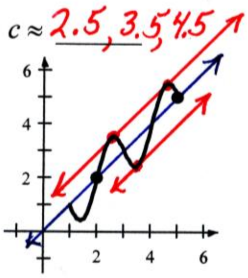



17. $c\approx$ 2.7, 4.3

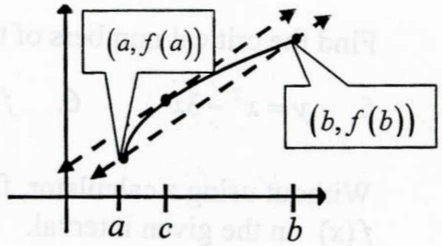



note: a third tangent line would not be on the (2,5) interval

18. $c\approx$ 2.5, 3.5, 4.5



**MEAN VALUE THEOREM**: If $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$, then there is a number $c$ in $(a,b)$




such that $\quad f'(c)=\dfrac{f(b)-f(a)}{b-a}$

tangent slope (inst. rt. of ch.)

secant slope (avg. rt. of ch.)

Informally: The Mean Value Theorem states that given the right conditions of continuity and differentiability, there will be at least one tangent line parallel to the secant line.
In still other words: The instantaneous rate of change (slope of tangent) will equal the average rate of change (slope of secant) at least once.

IROC = AROC

Example 19 Given $f(x)=3-\frac{6}{x}$, find all $c$ which satisfy the Mean Value Theorem on the interval $[3,6]$.

Points for secant line : (3,1) and (6,2)

$\text{AROC}=\dfrac{2-1}{6-3}=\dfrac{1}{3}$

$f(x)=3-6x^{-1}$

$f'(x)=6x^{-2}=\dfrac{6}{x^2}$

IROC

$\dfrac{6}{x^2}=\dfrac{1}{3}$

$18=x^2$

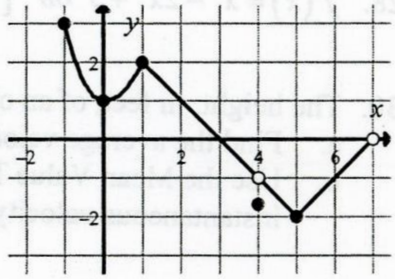
$x=\pm\sqrt{18}$

$c=\sqrt{18}$ or $3\sqrt{2}$

Note: $-\sqrt{18}$ is not in the interval.

**ASSIGNMENT** 4-1

Use the graph of $y=f(x)$ at the right for Problems 1-4.




1. What is the absolute maximum value of $f(x)$?
2. At what point does $f(x)$ reach a global minimum?
3. At what $x$-value(s) does $f(x)$ have a relative minimum?
4. At what $x$-value(s) does $f(x)$ have a relative maximum?